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M.Sc. (Sem. - II) Examination September - 2025

Mathematics (PGMTH : 2045) Special Functions-II

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions

(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the above details on your answer book
(2) Attempt all questions.
(3) Figures to the right indicate marks of the questions.
(4) Follow usual notations.

Seat No.:

Student's Signature

Q. 1 **Answer any two** **14**

(I) With usual notation prove that

$$P_n(\cos\alpha)=\left(\frac{\sin\alpha}{\sin\beta}\right)^n\sum_{k=0}^{n}C_{n,k}\left\{\frac{\sin(\beta-\alpha)}{\sin\alpha}\right\}^{n-k}P_k(\cos\beta)$$

(II) Prove that

$$\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}p_n(x)dx=2^{\alpha+\beta-1}B(\alpha+\beta)_3F_2\left[\begin{matrix}-n,n+\ 1;\beta; \\ 1,\alpha+\beta;\end{matrix}\ 1\right]$$

(III) With usual notation prove that

$$P_n(x)=\left(\frac{1+x}{2}\right)^n F\left[\begin{matrix}-n,-n; & \\ & \dfrac{x-1}{x+1} \\ 1; & \end{matrix}\right]$$

Q. 2 **Answer any two** **14**

(I) With usual notation prove that

$$\sum_{n=0}^{\infty}\frac{p_n(x)t^n}{(n!)^2}={}_0F_1\left[-;1;\frac{t}{2}(x+1)\right]{}_0F_1\left[-;1;\frac{t}{2}(x-1)\right]$$

(II) Prove that.

$$\int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1}$$

(III) For non negative integral n show that

$$x^n=\frac{n!}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(2n-4k+1)\,P_{n-2k}(x)}{k!\left(\frac{3}{2}\right)_{n-k}}$$

Q. 3 **Answer any two** **14**

(I) With usual notation prove that

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)\,t^n}{n\;!} = (1-4t^2)^{-\frac{1}{2}}\exp\left[y^2 - \frac{(y-2xt)^2}{1-4t^2}\right]$$

(II) Prove that

$$H_n(x)$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{{}_1F_1\left(-k; \frac{3}{2}+n-2k; 1\right)(-1)^k n!\,(2n-4k+1)\,P_{n-2k}(x)}{k!\left(\frac{3}{2}\right)_{n-2k}}$$

(III) With usual notation prove that

$$H_n(x) = 2^{n+1}\exp(x^2)\int_x^{\infty} \exp(-t^2)\,t^{n+1}\,P_n\left(\tfrac{x}{t}\right)\,dt$$

Q. 4 **Answer any two** **14**

(I) Prove that

$$\sum_{n=0}^{\alpha} \frac{H_{n+k}(x)t^n}{n!} = \exp(2xt-t^2)H_k(x-t)$$

(II) Prove that

(a) $x^n = \sum_{n=0}^{\infty} \dfrac{n!H_{n-2k}(x)}{2^n k!(n-2k)!}$

(b) $H_n(x) =$

$$(2x)^n\,{}_2F_0\left[\begin{matrix} -\frac{n}{2},\ \frac{-n+1}{2}; & \\ & \frac{-1}{(x)^2} \\ ---; & \end{matrix}\right]$$

(III) With usual notation prove that

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x)\,t^n}{n!}$$

$$= (1-2xt)^{-c}\,{}_2F_0\left[\begin{matrix} \frac{c}{2},\ \frac{c}{2}+\frac{1}{2}; & \\ & \frac{-4t^2}{(1-2xt)^2} \\ ---; & \end{matrix}\right]$$

Q. 5 **Answer any two** **14**

(I) If $\{\phi_n(x)\}$ is a simple set of polynomials and if $p(X)$ is a polynomial of degree m, then show that there exists constant c_k such that $p(X) = \sum_{k=0}^{m} c_k\phi_k(x)$, where c_k are functions of k and of any parameters involved in $p(X)$.

(II) State and prove the Christoffel-Darboux formula for orthogonal polynomials $\phi_n(x)$

(III) Prove that $\sum_{n=0}^{\infty} \dfrac{P_n(x)}{n!}t^n = e^{xt} J_0\left(t\sqrt{1-x^2}\right)$
